Household demand: linear expenditure system
Short Note

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1 Household utility maximisation

An average household chooses a consumption bundle that maximises utility subject to the budget constraint.

1.1 Bottom level

From the bottom level, with Q being the number of households, an average household chooses a consumption bundle, which consists of commodities $\frac{X_i^{total}}{Q}$, $i = 1, 2, 3...g$. Household consumption of commodity i is $X_i = \frac{X_i^{total}}{A_i Q}$, where A_i accounts for exogenous commodity-i-augmenting taste changes.

Household optimal decision-making:

$$\max U(X_1 ... X_g)$$

subject to

$$\underbrace{X_i = \underset{s=1,2}{CES}(X_{i,s})}_{\text{Source = 1.domestic source and 2. imports}} \quad \text{and} \quad \sum_{s=1}^{2} \sum_{i=1}^{g} P_{i,s} X_{i,s} = Y$$

- Household demand of commodity X_i is a result of a CES function of two imperfectly substitutable sources: domestic production and imports, denoted as $s = 1, 2$.
- The total household disposable income Y is the aggregate consumer budget that is to be allocated to across commodities X to maximise utility. Y is exogenous in terms of the household demand system, i.e., determined elsewhere or simply exogenous.

- $P_{i,s}$ is the consumer price of good i from source s. Note that consumer price of good i from source s may be different to the purchase price paid by other users, due to taxes and distribution cost etc.
- The representative household assumption could be changed to include different types of households if running very long run simulation allowing for substantial demographic changes or simulations involving major changes in income distributions.

We can combine the budget constraints and simplify household optimal decision-making setting:

$$\max U(X_1 ... X_g)$$
$$\text{subject to} \quad \sum_{i=1}^{g} P_i X_i = Y$$

1.2 LES

A Klein-Rubin utility function (also known as Stone-Geary function) is:

$$U = \prod (X_i - X_i^{sub})^{\beta_i}$$

or in its log form:

$$u = \sum_i \beta_i \ln(X_i - X_i^{sub})$$

- X_i^{sub} is the subsistence consumption of commodity i
- β_i is the share of supernumerary (or luxury) expenditure of good i in total supernumerary expenditure, sum across to 1 ($\sum_i \beta_i = 1$).

Klein-Rubin: one of the empirically least challenging specification in view of the resource constraint.

This makes the LES be a popular option for modeling consumer preferences in CGE models when data are scarce, as the underlying functional form is parsimonious in parameters.

1.3 Top level derivation

A top level household utility maximisation problem with a Klein-Rubin utility function is:

$$\max_{X_i} \sum_i \beta_i \ln(X_i - X_i^{sub}) \quad \text{s.t} \qquad \sum_i P_i X_i = Y \tag{1}$$

Solve for the First Order Condition, then we have the following **Linear Expenditure System**:

$$P_i X_i = P_i X_i^{sub} + \beta_i \big(Y - \sum_i P_i X_i^{sub}\big) \tag{2}$$

From Equation (2), it is clear that β_i is the **marginal budge share** of good i:

$$\frac{\partial (P_i X_i)}{\partial Y} = \beta_i \tag{3}$$

Write in Lagrange form:

$$L = \sum_i \beta_i \ln(X_i - X_i^{sub}) - \lambda(Y - \sum_i P_i X_i)$$

Solve for F.O.C:

$$\begin{aligned} \frac{\partial L}{\partial X_i} = 0 \quad &\rightarrow \qquad \beta_i = \lambda P_i (X_i - X_i^{sub}) \\ \sum_i \beta_i = 1 \quad &\rightarrow \qquad \sum_i \beta_i = 1 = \lambda \sum_i P_i (X_i - X_i^{sub}) \end{aligned}$$

$$\begin{aligned} \text{Combine} \Rightarrow \quad \beta_i &= \frac{P_i (X_i - X_i^{sub})}{\sum_i P_i (X_i - X_i^{sub})} \\ &= \frac{P_i (X_i - X_i^{sub})}{Y - \sum_i P_i X_i^{sub}} \quad \left(\frac{\text{Luxury spending on good } i}{\text{Total luxury spending}} \right) \end{aligned}$$

Rearrange

$$\Rightarrow \underbrace{P_iX_i}_{\text{Total expenditure on good } i} = \underbrace{P_iX_i^{sub}}_{\text{Subsistence expenditure on good } i} + \underbrace{\beta_i\big(Y - \sum_i P_iX_i^{sub}\big)}_{\text{Luxury spending on good } i\text{, which is a share } (\beta_i) \text{ of the total luxury spending } W^{lux} \text{ (} W3LUX \text{ in code)}} \qquad (2)$$

Define $W^{lux} = Y - \sum_i P_iX_i^{sub}$, Equation (2) becomes

$$P_iX_i = P_iX_i^{sub} + \beta_i W^{lux} \qquad (4)$$

This LES equation shows that the total expenditure on good i is a linear

Linearise Equation (4):

$$\begin{aligned} p_i + x_i &= S_i^{sub}(p_i + x_i^{sub}) + S_i^{lux} w^{lux} \\ x_i &= (1 - S_i^{lux})x_i^{sub} + S_i^{lux}(w^{lux} - p_i) \end{aligned} \qquad (5)$$

The linearised equation (5) shows that the % change of consumption of good i is a shared weighted sum of % change in subsistence consumption of good i and the % change in total luxury expenditure deflated by the price of good i. $S_i^{lux} = \dfrac{X_i^{lux}}{X_i}$ is the share of supernumerary expenditure in total expenditure of good i ($B3LUX$ in ORANI code).

It is easy to see that this demand equation is called "Linear Expenditure System" because the expenditure is linear in prices and income. The demand equations are clearly homogeneous of degree zero in prices and income. (There's no specific cross-price effects, only income effect in cross-price elasticities.)

When modelling, subsistence consumption is determined exogenously to the household demand system, and is typically assumed to be either fixed or to vary with the population size or the number of households.

Linearisation: three handy rules for deriving percentage-change (lower-case) forms of levels (upper-case)

- The Product rule: $X = \beta A * B \quad \rightarrow \quad x = a + b$
- The Power rule: $X = \beta A^{\gamma} \quad \rightarrow \quad x = \gamma a$
- The Sum rule: $X = A + B \quad \rightarrow \quad x = S_A * A + S_B * B$, where $S_A = \frac{A}{A+B}$ and $S_B = \frac{B}{A+B}$

Linearisation of budget constraint

Write the budget constraint $Y = \sum_i P_i X_i$ in linearised % form:

$$y = \frac{\Delta Y}{Y} = \sum_i \frac{P_i X_i}{Y}(p_i + x_i)$$

One more step to the theoretical derivation, we allow for taste changes towards good i by adding a subsistence-specific taste shifter (a_i^{sub}) and a luxury-specific taste shifters (a_i^{lux}) to Equation (5). These work like technical changes in production functions. We then have the extended LES equation :

$$x_i = (1 - S_i^{lux})(x_i^{sub} + a_i^{sub}) + S_i^{lux}(w^{lux} - p_i + a_i^{lux}) \tag{6}$$

2 Calibration of S_i^{lux}

Problem: We don't really know S_i^{sub} and S_i^{lux}, the subsistence and luxury shares of expenditure on good i.

Solution: Calibrate to find the initial values of S_i^{lux}

The luxury share of expenditure on good i can be written as a share of total expenditure of good i:

$$S_i^{lux} = \frac{P_i X_i - P_i X_i^{sub}}{P_i X_i} \tag{7}$$

Also define S_i as the **average budget share** of commodity i, i.e., the fraction of income spent on the good i.

$$S_i = \frac{P_i X_i}{\sum_{i=1}^{g} P_i X_i} = \frac{P_i X_i}{Y} \tag{8}$$

Luxury spending on good i can be represent by a function of β_i, the share of total luxury spending on good (i) (*W3LUX* in code), and S_i, the average budget share of commodity i.

Sub in Eq(2) $\beta_i = \frac{P_i(X_i - X_i^{sub})}{Y - \sum_i P_i X_i^{sub}}$, and Eq(8) to Eq(7).

$$S_i^{lux} = \frac{\frac{P_i(X_i - X_i^{sub})}{Y - \sum_i P_i X_i^{sub}}}{\frac{P_i X_i}{Y}} \cdot \frac{Y - \sum_i P_i X_i^{sub}}{Y}$$

$$= \underbrace{\frac{\beta_i}{S_i}}_{\text{(1) Expenditure elasticity for commodity } i \ (\epsilon_i)} \cdot \underbrace{\frac{Y - \sum_i P_i X_i^{sub}}{Y}}_{\text{(2) Ratio of luxury spending to total spending. This is the negative inverse of the \textbf{Frisch} parameter } = -\frac{1}{F}} \tag{9}$$

(1): ϵ_i (*EPS* in code): Expenditure elasticity for commodity i: Ratio of marginal budget share (Eq.3) to the average budget share.

- It measures the responsiveness of the quantity demanded for a good to a change in income.
- These values can be observed.

$$\epsilon_i = \frac{\beta_i}{S_i} = \frac{\frac{\partial(P_i X_i)}{\partial Y}}{\frac{P_i X_i}{Y}} = \frac{\frac{\Delta(P_i X_i)}{\Delta Y}}{\frac{P_i X_i}{Y}} = \frac{\frac{\Delta(P_i X_i)}{P_i X_i}}{\frac{\Delta Y}{Y}} = \frac{p_i + x_i}{y} \rightarrow \frac{\text{\% change in price and quanity of commodity } i}{\text{\% change in total income}}$$

(2): Ratio of luxury spending to total spending, which is the negative inverse of the **Frisch**

Parameter, F.

$$F = -\frac{1}{\frac{Y - \sum_i P_i X_i^{sub}}{Y}}$$

The Frisch parameter is the income elasticity of the marginal utility of income, measuring sensitivity of the marginal utility of income to income/total expenditures. It is also called money flexibility, establishing a relationship between own-price and income elasticities.

Derivation of the Frisch parameter:

The marginal utility of income: $\lambda = \frac{\partial U}{\partial Y}$

From the budget-constraint maximum,

$$\begin{aligned}\lambda &= \frac{\beta_i}{P_i(X_i - X_i^{sub})} \\ &= \frac{1}{Y - \sum\limits_i P_i X_i^{sub}} > 0\end{aligned}$$

The income elasticity of the marginal utility of income, known as the **Frisch parameter** is the negative inverse of the supernumerary ratio:

$$F = \frac{\frac{\partial \lambda}{\partial Y}}{\frac{\lambda}{Y}} = \frac{\partial \lambda}{\partial Y}\frac{Y}{\lambda} = -\frac{Y}{Y - \sum_i P_i X_i^{sub}} = -\frac{1}{\frac{Y - \sum_i P_i X_i^{sub}}{Y}} < 0$$

Hence we see that the marginial utility of income is positive but declines with increasing income.

A higher ratio of luxury spending leads to a smaller Frisch parameter in absolute value $|F|$.

Hence the luxury share of expenditure on commodity i (Eq.9) becomes:

$$S_i^{lux} = \frac{X_i^{lux}}{X_i} = -\frac{\epsilon_i}{F}$$

The final LES demand system (Eq.6) is written as:

$$x_i = (1 - S_i^{lux})(x_i^{sub} + a_i^{sub}) + S_i^{lux}(w^{lux} - p_i + a_i^{lux}) \quad (6)$$

Where

$$S_i^{lux} = -\frac{\epsilon_i}{F}$$

Both ϵ_i (EPS) and the Frisch parameter can be estimated empirically.

Also note the linear relationship between value of expenditure on commodity i and the income/total expenditure

$$w_i = p_i + x_i = \epsilon_i \cdot y$$